1.1 Recursively Defined Sequences

In this lesson you will

- write **recursive** definitions and formulas for patterns and sequences
- learn to recognize and write formulas for **arithmetic** and **geometric** sequences
- use recursion to solve problems related to economics and fractal geometry

Many mathematical patterns can be described using the idea of recursion. **Recursion** is a process in which each step of a pattern is dependent on the step or steps that come before it. Example A in your book involves a pattern that can be defined recursively. Read that example carefully, and then read the example below.

EXAMPLE

Each square in this pattern has side length 1 unit. Imagine that the pattern continues. Find the perimeter of Figure 9. Which figure has a perimeter of 76 units? Write a recursive definition to find the perimeter of any figure in the pattern.
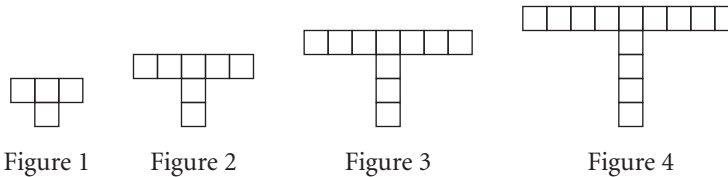

Figure 1   Figure 2   Figure 3   Figure 4

▶ **Solution**

The perimeters of Figures 1–4 are 10, 16, 22, and 28, respectively. Notice that with each new figure, the perimeter increases by 6. You can record the information for the given figures in a table and then continue the pattern to find that Figure 9 has a perimeter of 58 units and that Figure 12 has a perimeter of 76 units.

Figure	1	2	3	4	5	6	7	8	9	10	11	12
Perimeter	10	16	22	28	34	40	46	52	58	64	70	76

Here is a recursive definition for the pattern. It states the starting value and tells how to calculate each subsequent value:

perimeter of Figure 1 = 10

perimeter of Figure n = perimeter of Figure $(n - 1) + 6$

Read the information between Examples A and B in your book. Notice that the perimeters in the example above represent this **sequence:**

10, 16, 22, 28, . . .

Here is a **recursive formula** for the sequence:

$u_1 = 10$
$u_n = u_{n-1} + 6$ where $n \geq 2$

This means *the first term is 10* and *each subsequent term is equal to the previous term plus 6.*

(continued)

Example B in your book presents another example of a recursive sequence. Work through this example on your own. In the sequences you have seen so far, each term is generated by *adding* a fixed number to the previous term. For example, in the sequence of perimeters, 6 is added to each term to get the next term. Sequences like this are called **arithmetic sequences,** and the number that is added is called the **common difference** (because it is the *difference* between each term and the previous term). Read the definition of **arithmetic sequence** in your book.

Investigation: Monitoring Inventory

In your book, read the first paragraph of the investigation. You can use spreadsheet software or your calculator to model what happens to the number of unmade prints and the inventory at the two stores. See Calculator Note 1C for instructions about working with a spreadsheet on your calculator. If you are not able to use a spreadsheet, you can use recursion on your calculator to complete the table.

The first column shows the number of months. Fill down to show 30 months. Each month, Art supplies FineArt with 40 prints and Little Print Shoppe with 10 prints. That means the number of unmade prints will decrease by 50 each month. Create a formula for the Unmade column and fill down.

	A month	B unmade	C	D	E
1	1	2000			
2	2	1950			
3	3	1900			
4	4	1850			
5	5	1800			
B2	=b1−50				

Create formulas and fill down to model the number of prints that FineArt and Little Print Shoppe will have. Then use the spreadsheet to answer the questions in Step 2.

To answer question a, scroll down until you find the month when the number of unmade prints is equal to or less than the number of prints at FineArt. Refer to the spreadsheet at right.

To answer question b, find the month in which the inventory of prints at FineArt is greater than twice the number of unmade prints. You should find this occurs in month 27.

	A month	B unmade	C fineart	D little
14	14	1350	990	130
15	15	1300	1030	140
16	16	1250	1070	150
17	17	1200	1110	160
18	18	1150	1150	170
A18	=a17+1			

You have looked at arithmetic sequences, in which each term is generated by adding a fixed number to the previous term. In a **geometric sequence,** each term is generated by *multiplying* the previous term by a constant. For example,

3, 6, 12, 24, 48, . . .

is a geometric sequence in which each term is 2 times the previous term. The rule for this sequence is

$u_1 = 3$
$u_n = 2 \cdot u_{n-1}$ where $n \geq 2$

The constant by which the terms in a geometric sequence are multiplied is called the **common ratio** because it is the ratio of each term to the previous term. In the sequence above, $\frac{6}{3} = \frac{12}{6} = \frac{24}{12} = \frac{48}{24} = 2$.

Example C in your book presents a geometric sequence related to a **fractal.** Work through this example and then read the rest of the lesson.

1.2 Modeling Growth and Decay

In this lesson you will

- explore geometric sequences that model **growth** and **decay**

In the previous lesson, you looked at several examples of *geometric sequences.* Recall that in a geometric sequence, each term is equal to the previous term times a *common ratio.* The recursive rule for a geometric sequence is in the form $u_n = r \cdot u_{n-1}$. In this lesson you will explore more geometric sequences.

In most of the sequences you have worked with, you have used u_1 as the first term. In some cases, it is more meaningful to consider the first term to be the zero term, or u_0. The zero term represents the starting value before any change occurs.

Example A in your book uses a recursive formula to model the decreasing value of a car. Work through Example A and then read the example below.

EXAMPLE | TV Central is going out of business in 8 weeks. Each week until it closes, the company plans to reduce the prices from the previous week by 15%. A television is currently priced at $899. Write a recursive formula to find the price of the television after 8 weeks if it remains unsold.

▶ **Solution** | Each week, the price of the television will be 85% of the price from the previous week. Let u_0 represent the starting price of the television so that u_1 represents the value after 1 week and so on. The recursive formula that generates the sequence of weekly prices is

$$u_0 = 899$$
$$u_n = 0.85 \cdot u_{n-1} \quad \text{where } n \geq 1$$

You can use recursion on your calculator, as shown in Example A in your book, or you can use a spreadsheet to find u_8.

A week	B price	C	D	E	F
4	469.284				
5	398.891				
6	339.057				
7	288.199				
8	244.969				

B9   =b8·.85

After 8 weeks, the price of the television will be $244.97.

Investigation: Looking for the Rebound

If you drop a ball and let it bounce repeatedly, the rebound height becomes smaller with each bounce. The pattern of rebound heights can be modeled by a geometric sequence.

If you have access to a motion sensor, perform the experiment described in Step 1 of the investigation in your book. Your calculator will make a graph with the time since you pressed the trigger on the x-axis and the height of the ball on the y-axis.

(continued)

Trace the graph to find the initial height and the height of each rebound. (These heights correspond to the high points on the graph.) Make a table showing the bounce number and the rebound height. If you are not able to collect your own data, use the sample data below.

Bounce number	0	1	2	3	4	5	6
Rebound height (in.)	100	79	62	50	38	31	24

Graph your data on paper and on a calculator. At right is a graph of the sample data.
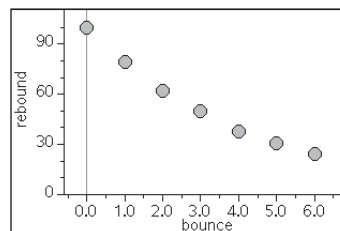
Calculate the rebound ratio for consecutive bounces.

$$rebound\ ratio = \frac{rebound\ height}{previous\ rebound\ height}$$

The rebound ratios for the sample data (rounded to the nearest hundredth) are 0.79, 0.78, 0.81, 0.76, 0.82, and 0.77.

Decide on a single value that best represents the rebound ratio for your ball. For the sample data, we'll use the mean, 0.79.

Use the rebound ratio you chose to write a recursive formula that models your sequence of rebound heights. Using an initial height of 100 inches and a rebound ratio of 0.79, the formula is

$u_0 = 100$
$u_n = 0.79 \cdot u_{n-1}$ where $n \geq 1$

You can generate the first six terms on your calculator.

Compare your experimental data with the terms generated by your formula, and think about the questions in Step 6 in your book.

In a recursive formula for a geometric sequence, you may find it helpful to think of the common ratio as 1 plus or minus a percent change. So, in place of r, you could write $(1 + p)$ or $(1 - p)$. For example, you could write the recursive rule for the TV Central example as $u_n = (1 - 0.15)u_{n-1}$, indicating that the price decreases by 15% each week.

Both the TV Central example and the rebound-height experiment involve **decay,** or geometric sequences that decrease. Example B in your book is about earning compound interest on a savings account. This is an example of **growth,** or a geometric sequence that increases. Work through Example B and then read the rest of the lesson. Note that if Gloria's account had earned 7% interest compounded *monthly,* then the recursive formula would have been

$u_0 = 2000$
$u_n = \left(1 + \frac{0.07}{12}\right)u_{n-1}$ where $n \geq 1$

In this formula, n is the number of months, not the number of years. Use this formula to compute the balance after a year (12 months). How does it compare with the balance when the interest is compounded yearly?

1.3 | A First Look at Limits

In this lesson you will

- investigate sequences that approach a **limit** in the long run

Some sequences increase or decrease without bound. Others have terms that approach a specific value, or *limit,* in the long run. In this lesson you will explore sequences that have limits.

Investigation: Doses of Medicine

Our kidneys continuously filter our blood, removing impurities. Doctors take this into account when prescribing the dosage and frequency of medicine. This investigation simulates what happens in the body when a patient takes medicine.

If you have the materials listed in your book, perform the experiments described. If not, just think about how the quantity of medicine would change. Try to work through the investigation on your own before reading the following text.

The first experiment starts with 1 L of liquid. Of this 1 L, 16 mL is tinted liquid, representing the medicine, and the rest is clear water, representing the blood.

Step 1 Suppose a patient's kidneys filter out 25% of the medicine each day. To simulate this, you can remove $\frac{1}{4}$, or 250 mL, of the mixture and replace it with 250 mL of clear water. You can repeat this many times to simulate what happens over several days. This table shows the amount of medicine in the bloodstream each day.

Day	Amount of medicine (mL)
0	16
1	12
2	9
3	6.75
4	5.0625

**Step 2** The recursive formula that generates this sequence is

$$u_0 = 16$$
$$u_n = (1 - 0.25)u_{n-1} \quad \text{where } n \geq 1$$

Step 3 You can use a spreadsheet or recursion on your calculator to generate more values. You should find that in 10 days, the amount of medicine in the blood will be less than 1 mL.

Step 4 Theoretically, the medicine will never be completely removed from the blood. Each day, the amount is multiplied by 0.75, giving a smaller and smaller quantity. However, this quantity will never be 0 because there is no number $x > 0$ such that $0.75x = 0$.

Step 5 This graph shows what happens in the long run. The amount of medicine decreases rapidly at first, but the decrease slows down and levels off near 0 mL.
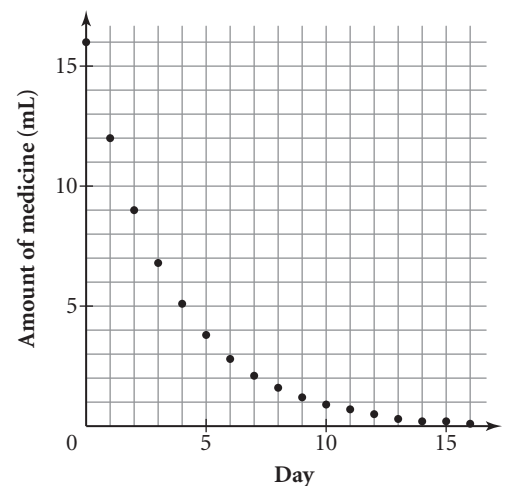
(continued)

Doctors sometimes prescribe regular doses of medicine to produce and maintain a certain level of medicine in the body. You can modify the experiment to simulate what happens when a patient takes medicine daily.

This experiment also starts with 1 L of liquid. Of this 1 L, 16 mL is tinted liquid, representing the medicine, and the rest is clear water, representing the blood.

Step 6   As before, assume the patient's kidneys filter out 25% of the medicine each day. This time, however, assume the patient takes another 16 mL dose each day. To simulate this situation, you can remove 250 mL of liquid and replace it with 234 mL of water and 16 mL of tinted liquid. This table shows the amount of medicine in the bloodstream each day.

Day	Amount of medicine (mL)
0	16
1	28
2	37
3	43.75
4	48.8125

**Step 7**   The recursive formula that generates this sequence is

$u_0 = 16$
$u_n = (1 - 0.25)u_{n-1} + 16$ where $n \geq 1$

Step 8   If you use your calculator to generate many terms of the sequence, you will find that they begin to level off at around 64 mL. So the liquid in the bowl will never turn into pure medicine.

Step 9   This graph shows what happens to the level of medicine in the blood after many days. The amount increases rapidly at first, but the increase slows down and levels off near 64 mL.
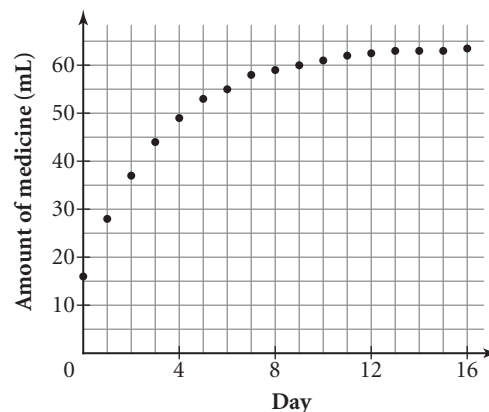
The elimination of medicine from the human body is an example of a dynamic, or changing, system. Dynamic systems often reach a point of stability in the long run. For example, in the second experiment, the amount of blood increased quickly at first but eventually stabilized. The quantity associated with a point of stability, for example, 64 mL in the second experiment, is called a **limit.** Mathematically, we say that the sequence of numbers associated with the system approaches that limit.

The sequence from the first experiment had a limit of zero. The sequence from the second experiment was shifted and approached a nonzero value. A **shifted geometric sequence** includes an added term in the recursive rule.

The example in your book involves another shifted geometric sequence. Work through this example carefully.

Graphing Sequences

In this lesson you will

- explore relationships among different **representations of sequences**
- find sequences that **model data**

You can graph the terms of a sequence. For example, this graph represents the sequence generated by the recursive formula $u_0 = 3$ and $u_n = u_{n-1} - 1$ where $n \geq 1$.

This graph appears to be **linear.** That is, the points appear to lie on a line. The common difference, $d = -1$, makes each new point fall 1 unit below the previous point.

The graph of a sequence is a **discrete graph,** meaning that it is made of isolated points. It does not make sense to connect the points in the graph of a sequence with a continuous line or curve because the term number, n, must be a whole number.
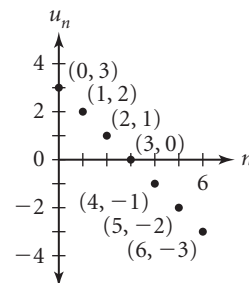
Investigation: Match Them Up

The investigation in your book shows representations of sequences in table, formula, and graph form. Match each table with the formula and graph that represent the same sequence. Try to complete the investigation on your own before reading further.

Below are the correct matches, along with explanations of the reasoning you might use to make the matches.

1, B, iv: The starting value is 8, so the correct formula must be A or B. Formula A gives $u_1 = 6$, which does not match the value in the table. The first few terms generated by formula B are 8, 4, 2, and 1, which match the values in the table. Formula B represents a decreasing sequence that starts at 8 and gets smaller and smaller, approaching 0. Graph iv shows a sequence that fits this description.

2, C, vi: The sequence starts at 0.5, and each subsequent term is 2 times the previous term. This matches formula C. The graph should start at 0.5 and increase at a faster and faster rate. The correct graph must be vi.

3, F, iii: The sequence has starting value of -2. It does not have a constant difference or a constant ratio, so it is not purely geometric or arithmetic. The only formula fitting this description is F. Graphs iii and v have a starting value of -2. Graph v is linear and so represents an arithmetic sequence. Graph iii is the correct graph.

4, D, v: The sequence has a starting value of -2, so the correct formula must be D or F. Using formula D to generate the first seven terms gives -2, 0, 4, 6, 8, 10, 12. The values of terms 0, 2, 5, and 7 match those in the table, so formula D matches table 4. Graph v starts at -2 and rises by 2 units with each point. This matches formula D.

5, A, ii: The sequence has a starting value of 8, so the correct formula must be A or B. Using formula A to generate the first eight terms gives 8, 6, 4, 2, 0, -2, -4, -6. Terms 0, 1, 3, 5, and 7 match those in the table. Graph ii starts at 8 and falls by 2 units with each point. This matches formula A.

(continued)

6, E, i: The sequence has a starting value of -4, and each term is the same as the previous term (i.e., the sequence has a constant difference of 0). Formula E generates this sequence. Graph i, which shows a sequence in which every term is -4, matches the table and formula.

Here are some general observations about the relationships between sequences, recursive formulas, and graphs.

- Arithmetic sequences have constant differences, recursive rules in the form $u_n = u_{n-1} + d$, and linear graphs with a slope equal to the constant difference. If the constant difference is positive, the sequence increases and the graph slopes upward. If the constant difference is negative, the sequence decreases and the graph slopes downward.

- Geometric sequences have constant ratios, recursive rules in the form $u_n = ru_{n-1}$, and curved graphs. If the constant ratio is greater than 1, the sequence increases and the graph grows slowly at first and then more quickly. If the constant ratio is between 0 and 1, the sequence decreases and the graph decreases quickly at first and then more slowly, approaching 0.

- Shifted geometric sequences have neither a constant difference nor a constant ratio. They have recursive rules in the form $u_n = au_{n-1} + b$ and curved graphs.

Sometimes real-world data can be **modeled** with an arithmetic or geometric sequence. The shape of the graph can help you determine which, if either, type of sequence might fit. The example in your book models a set of data with a geometric sequence. Work through this example carefully, read the rest of the lesson, and then read the example below.

EXAMPLE | Aaron kept track of the number of math problems he solved each night last week and the length of time it took him to finish each assignment. Find a sequence model that approximately fits his data.

Problems	0	3	5	6	8	10	15
Time (min)	0	21	35	43	60	72	100

▶ **Solution** | The graph at top right is approximately linear, so an arithmetic sequence would be a good model.

The starting value is 0. To find the constant difference, find the slope between consecutive pairs of values. These slopes are 7, 7, 8, 8.5, 6, and 5.6, respectively. Try using the median slope, 7, as the constant difference. The model is then

$$u_0 = 0$$
$$u_n = u_{n-1} + 7 \quad \text{where } n \geq 1$$

Add those points to your graph as shown at bottom right.

The model appears to be a good fit.

(*problems,time*)

(*n,seq_model*)

(*problems,time*)
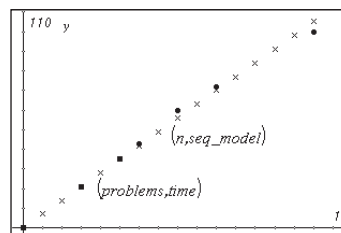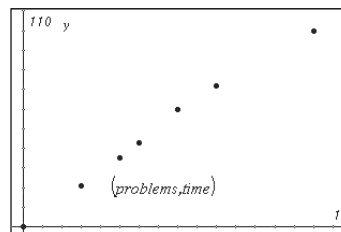
1.5 Loans and Investments

In this lesson you will

- use what you have learned about recursive sequences to calculate the monthly payment on a loan
- use recursion to calculate the **balance** in an **investment** account

Recursive formulas are useful for solving problems involving finances.

Investigation: Life's Big Expenditures

In this investigation you will use recursion to explore loan balances and payment options. Try to complete the steps on your own before reading the following text.

Part 1

Suppose you borrow $22,000 to buy a new car, to be paid back over 5 years (60 months). The bank charges interest at an annual rate of 7.9%, compounded monthly.

Step 1 The monthly interest rate is the annual interest rate divided by 12, which is $\frac{7.9\%}{12}$, or $\frac{0.079}{12}$. So, the first month's interest on $22,000 is $22,000$\left(\frac{0.079}{12}\right)$, or $144.83. The total balance is then $22,144.83. If you pay $300 at the end of the month, the remaining balance will be $21,844.33.

Step 2 The recursive rule for the sequence of monthly balances is

$$u_0 = 22{,}000$$
$$u_n = \left(1 + \frac{0.079}{12}\right)u_{n-1} - 300 \quad \text{where } n \geq 1$$

You can use your calculator to find the balances for the first 6 months. Refer to Calculator Notes 1I and IJ for instructions on creating and graphing sequences.

The calculator table shows that the balance will be paid off in 101 months, or 8 years and 5 months.

(continued)

Step 3 Experiment with other payment amounts. You should find that $445.03 is the smallest monthly payment that allows you to pay off the loan in exactly 60 months.

A month	B month60	C	D	E
◆	=seqn((1+.079/12			
1	0.	22000.		
2	1.	21699.8		
3	2.	21397.6		
4	3.	21093.5		

B   month60:=seqn$\left(\left(1+\dfrac{.079}{12}\right)\cdot u(n-1)-445.03,\right.$▸

A month	B mont...	C	D	
◆	=seqn((1+			
59	58.	881.241		
60	59.	442.013		
61	60.	-.107477		
62	61.	-445.138		

B60   =$\left(1+\dfrac{.079}{12}\right)\cdot b59-445.03$

Step 4 If you pay $445.03 per month, you will make 59 payments of $445.03, plus one payment of $444.92. This is the normal monthly payment of $445.03, minus the $0.11 overpayment, which you can see from the table above right. (You cannot use the balance at 59 months because you would still owe interest on it.) Therefore, the total amount you actually pay for the car is 59($445.03) + $444.92 = $26,701.69.

Part 2

To find the monthly payment for a 30-year (360-month) mortgage of $146,000 with an annual interest rate of 7.25%, compounded monthly, use your calculator to experiment with rules of the form

$u_0 = 146,000$

$u_n = \left(1 + \dfrac{0.0725}{12}\right)u_{n-1} - payment$ where $n \geq 1$

Try to find the smallest possible value for *payment* such that $u_{359} > 0$ and $u_{360} \leq 0$. You should find that the payment will be $995.98.

A monthn	B mortgage	C	D	E
◆ =seqn(u(n	=seqn((1+.0725/12			
1	0.	146000.		
2	1.	145886.		
3	2.	145772.		
4	3.	145656.		

B   mortgage:=seqn$\left(\left(1+\dfrac{.0725}{12}\right)\cdot u(n-1)-995.9\right.$▸

	monthn	B mortgage	C	D	E
◆	seqn(u(n	=seqn((1+.0725/12			
358	357.	2948.88		
359	358.	1970.72		
360	359.	986.646		
361	360.	-3.37281		

B361   =$\left(1+\dfrac{.0725}{12}\right)\cdot b360-995.98$

The total amount you would actually pay for the house is $995.98 · 359 + $992.61 = $358,549.43.

The example in your book presents a problem about investments. Work through this problem on your own.